INVERSE FUNCTIONS
ALGEBRA 2 WITH TRIGONOMETRY

The idea of inverses, or opposites, is very important in mathematics. So important, in fact, that the word is used in many different contexts, including the additive and multiplicative inverses of a number. The actions of certain functions can be reversed as well. The rules governing the reversal themselves can be functions.

Exercise #1: Consider the two linear functions given by the formulas \( f(x) = \frac{3x+7}{2} \) and \( g(x) = \frac{2x-7}{3} \).

(a) Calculate \( f(5) \) and \( g(11) \).

\[
\begin{align*}
  f(5) &= \frac{3(5)+7}{2} = \frac{15+7}{2} = \frac{22}{2} = 11 \\
  g(11) &= \frac{2(11)-7}{3} = \frac{22-7}{3} = \frac{15}{3} = 5
\end{align*}
\]

(b) Calculate \( f(0) \) and \( g\left(\frac{7}{2}\right) \).

\[
\begin{align*}
  f(0) &= \frac{3(0)+7}{2} = \frac{0+7}{2} = \frac{7}{2} \\
  g\left(\frac{7}{2}\right) &= \frac{2(7/2)-7}{3} = \frac{7-7}{3} = \frac{0}{3} = 0
\end{align*}
\]

(c) Calculate \( f(-3) \) and \( g(-1) \).

\[
\begin{align*}
  f(-3) &= \frac{3(-3)+7}{2} = \frac{-9+7}{2} = \frac{-2}{2} = -1 \\
  g(-1) &= \frac{2(-1)-7}{3} = \frac{-2-7}{3} = \frac{-9}{3} = -3
\end{align*}
\]

(d) Calculate \( f(g(5)) \).

\[
\begin{align*}
  g(5) &= \frac{2(5)-7}{3} = \frac{10-7}{3} = \frac{3}{3} = 1 \\
  f(1) &= \frac{3(1)+7}{2} = \frac{3+7}{2} = \frac{10}{2} = 5
\end{align*}
\]

(e) Without calculation, determine the value of \( f\left(\frac{\pi}{2}\right) \).

\( f\left(\frac{\pi}{2}\right) = \pi \)

The two functions seen in Exercise #1 are inverses because they literally “undo” one another. The general idea of inverses, \( f(x) \) and \( g(x) \), is shown below in the mapping diagram.

Exercise #2: If the point \((-3, 5)\) lies on the graph of \( y = f(x) \), then which of the following points must lie on the graph of its inverse?

1. \((-3, 5)\)
2. \((-5, 3)\)
3. \((5, -3)\)
4. \(\left(-\frac{1}{3}, \frac{1}{5}\right)\)

Since \( f \) maps an input of -3 to an output of 5, its inverse must do the opposite, i.e. map an input of 5 to an output of -3.
Inverse functions have their own special notation. It is shown in the box below.

**Inverse Function Notation**

If a function \( y = f(x) \) has an inverse that is also a function we represent it as \( y = f^{-1}(x) \).

**Exercise #3:** The linear function \( f(x) = \frac{2}{3}x - 2 \) is shown graphed below. Use its graph to answer the following questions.

(a) Evaluate \( f^{-1}(2) \) and \( f^{-1}(-4) \).

Since \( f(6) = 2 \) then \( f^{-1}(2) = 6 \).
Since \( f(-3) = -4 \) then \( f^{-1}(-4) = -3 \).

(b) Determine the \( y \)-intercept of \( f^{-1}(x) \).

Since \( f(3) = 0 \) then \( f^{-1}(0) = 3 \).
Hence the \( y \)-intercept is 3.

(c) On the same set of axes, draw a graph of \( y = f^{-1}(x) \).

Simply take each point and switch its \( x \) and \( y \) coordinates to generate the graph of the inverse.

**Exercise #4:** A table of values for the simple quadratic function \( f(x) = x^2 \) is given below along with its graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) Graph the inverse by switching the ordered pairs.

<table>
<thead>
<tr>
<th>( x )</th>
<th>4</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f^{-1}(x) )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) What do you notice about the graph of this function’s inverse?

It is not a function itself. This is easily seen by the fact that it fails the Vertical Line Test.

**Existence of Inverse Functions**

A function will have an inverse that is also a function if and only if it is one-to-one. Hence, a quick way to know if a function has an inverse that is also a function is to apply the Horizontal Line Test.
Inverse Functions

Algebra 2 with Trigonometry – Homework

Skills

1. If the point $(-7, 5)$ lies on the graph of $y = f(x)$, which of the following points must lie on the graph of its inverse?

   (1) $(5, -7)$  
   (2) $\left(-\frac{1}{7}, \frac{1}{5}\right)$  
   (3) $(7, -5)$  
   (4) $\left(\frac{1}{7}, -\frac{1}{5}\right)$

   Remember, the key to inverses is that the input and output simply switch roles. So, $y$ becomes $x$ and $x$ becomes $y$.

2. The function $y = f(x)$ has an inverse function $y = f^{-1}(x)$. If $f(a) = -b$ then which of the following must be true?

   (1) $f^{-1}(-b) = -a$  
   (2) $f^{-1}\left(\frac{1}{a}\right) = -\frac{1}{b}$  
   (3) $f^{-1}(-b) = a$  
   (4) $f^{-1}(b) = -a$

   Again, input becomes output and output becomes input. Thus, choice (3).

3. The graph of the function $y = g(x)$ is shown below. The value of $g^{-1}(2)$ is

   (1) 2.5  
   (2) -4  
   (3) 0.4  
   (4) -1

   To evaluate $g^{-1}(2)$ we must go to a $y$-value of 2 and find the corresponding $x$-value, which in this case is -1.

4. Which of the following functions would have an inverse that is also a function?

   (1)  
   (2)  
   (3)  
   (4) 

5. For a one-to-one function it is known that $f(0) = 6$ and $f(8) = 0$. Which of the following must be true about the graph of this function’s inverse?

   (1) its $y$-intercept = 6  
   (2) its $y$-intercept = 8  
   (3) its $x$-intercept = -6  
   (4) its $x$-intercept = -8

   Since $f^{-1}(0) = 8$, by definition, the $y$-intercept of the inverse must be 8.
6. The function \( y = h(x) \) is entirely defined by the graph shown below.

(a) Sketch a graph of \( y = h^{-1}(x) \). Create a table of values if needed.

(b) Write the domain and range of \( y = h(x) \) and \( y = h^{-1}(x) \) using interval notation.

\[
\begin{align*}
y &= h(x) \\
y &= h^{-1}(x)
\end{align*}
\]

Domain: \([-5, 4]\) \hspace{1cm} Domain: \([-4, -1]\)

Range: \([-4, -1]\) \hspace{1cm} Range: \([-5, 4]\)

APPLICATIONS

7. The function \( y = A(r) = \pi r^2 \) is a one-to-one function that uses a circle’s radius as an input and gives the circle’s area as its output. Selected values of this function are shown in the table below.

<table>
<thead>
<tr>
<th>( A(r) )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>( \pi )</td>
<td>( 4\pi )</td>
<td>( 9\pi )</td>
<td>( 16\pi )</td>
<td>( 25\pi )</td>
<td>( 36\pi )</td>
</tr>
</tbody>
</table>

(a) Determine the values of \( A^{-1}(9\pi) \) and \( A^{-1}(36\pi) \) from using the table.

\[
\begin{align*}
A^{-1}(9\pi) &= 3 \\
A^{-1}(36\pi) &= 6
\end{align*}
\]

(b) Determine the values of \( A^{-1}(100\pi) \) and \( A^{-1}(225\pi) \) from using the table.

\[
\begin{align*}
A^{-1}(100\pi) &= 10 \\
A^{-1}(225\pi) &= 15
\end{align*}
\]

(c) The original function \( y = A(r) \) converted an input, the circle’s radius, to an output, the circle’s area. What are the inputs and outputs of the inverse function?

Input: The Circle’s Area \hspace{1cm} Output: The Circle’s Radius

REASONING

8. The domain and range of a one-to-one function, \( y = f(x) \), are given below in set-builder notation. Give the domain and range of this function’s inverse also in set-builder notation.

\[
\begin{align*}
y &= f(x) \\
y &= f^{-1}(x)
\end{align*}
\]

Domain: \( \{x \mid -3 \leq x < 5\} \) \hspace{1cm} Domain: \( \{x \mid x > -2\} \)

Range: \( \{y \mid y > -2\} \) \hspace{1cm} Range: \( \{y \mid -3 \leq y < 5\} \)